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SOME RELIABILITY APPLICATIONS OF THE VARIABILITY ORDERING

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ABSTRACT

The random variable X is said to be more variable than Y if
$E[f(X)] > E[f(Y)]$ for all increasing convex functions f . We prove
a preservation, under random sized sums, property of this ordering
and then applying it to branching processes and shock models. Other
applications of this ordering--to a population survival and to a
Poisson shock model--are also given.

SOME RELIABILITY APPLICATIONS OF THE VARIABILITY ORDERING

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1. A VARIABILITY RESULT

If X_1 and X_2 are random variables having respective distributions F_1 and F_2 , then we say that $X_1 \underset{v}{\leq} X_2$ (read X_1 is less variable than X_2) or equivalently that $F_1 \underset{v}{\leq} F_2$ if

$$\int_{-\infty}^{\infty} f(x) dF_1(x) \leq \int_{-\infty}^{\infty} f(x) dF_2(x)$$

for all increasing convex functions f . Some easily derived properties of this ordering are

1. $F_1 \underset{v}{\leq} F_2$ if and only if

$$\int_{a}^{\infty} \bar{F}_1(x) dx \geq \int_{a}^{\infty} \bar{F}_2(x) dx \quad \text{for all} \quad a$$

   where $\bar{F}_1 = 1 - F_1$.

2. If $F_i \underset{v}{\leq} G_i$, $i = 1,2$, then $F_1 * F_2 \underset{v}{\leq} G_1 * G_2$ where $*$ denotes convolution.

We will now present a theorem concerning this ordering and in Sections 2 and 3 apply it to branching processes and shock models. Other applications of the variability ordering to population survival models (Section 4) and to Poisson shock models (Section 5) will then be presented.

Theorem 1:

Let $X_1, X_2, \ldots$ be a sequence of nonnegative independent and identically distributed random variables and similarly $Y_1, Y_2, \ldots$. Let N and M be integer valued nonnegative random variables that are independent of the $\{X_i\}$ and $\{Y_i\}$ sequences. Then

$$X_i \underset{v}{\geq} Y_i \ , \ i \geq 1 \ , \ N \underset{v}{\geq} M \Rightarrow \sum_{i=1}^{N} X_i \underset{v}{\geq} \sum_{i=1}^{M} Y_i \ .$$

Proof:

We will first show that

$$\sum_{i=1}^{N} X_i \underset{v}{\geq} \sum_{i=1}^{M} X_i \ .$$

Let h denote an increasing convex function. To prove the above we must show that

$$(1) \qquad E\left[h\left(\sum_{1}^{N} X_i\right)\right] \geq E\left[h\left(\sum_{1}^{M} X_i\right)\right] \ .$$

Since $N \underset{v}{\geq} M$, and they are independent of the X_i , the above will follow if we can show that the function $g(n)$, defined by

$$g(n) = E[h(X_1 + \ldots + X_n)]$$

is an increasing convex function of n . As it is clearly increasing since h is and each X_i is nonnegative it remains to show that g is convex, or, equivalently, that

$$(2) \qquad g(n + 1) - g(n) \quad \text{is increasing in } n \ .$$

To prove this let $S_n = \sum_1^n X_1$, and note that

$$g(n + 1) - g(n) = E[h(S_n + X_{n+1}) - h(S_n)] .$$

Now,

$$E[h(S_n + X_{n+1}) - h(S_n) \mid S_n = t] = E[h(t + X_{n+1}) - h(t)]$$
$$= f(t) \quad \text{(say)}.$$

As h is convex, it follows that $f(t)$ is increasing in t . Also, as S_n increases in n , we see that $E[f(S_n)]$ increases in n . But

$$E[f(S_n)] = g(n + 1) - g(n)$$

and thus (2) and (1) are satisfied.

We have thus proven that

$$\sum_1^N X_1 \geq_v \sum_1^M X_1$$

and the proof will be completed by showing that

$$\sum_1^M X_1 \geq_v \sum_1^M Y_1$$

or, equivalently, that for increasing, convex h

$$E\left[h\left(\sum_1^M X_1\right)\right] \geq E\left[h\left(\sum_1^M Y_1\right)\right] .$$

But

$$E\left[h\left(\sum_{1}^{M} X_i\right) \mid M = m\right] = E\left[h\left(\sum_{1}^{m} X_i\right)\right] \quad \text{by independence}$$

$$\geq E\left[h\left(\sum_{1}^{m} Y_i\right)\right] \quad \text{since} \quad \sum_{1}^{m} X_i \underset{v}{\geq} \sum_{1}^{m} Y_i$$

$$= E\left[h\left(\sum_{1}^{M} Y_i\right) \mid M = m\right]$$

and the result follows by taking expectations of both sides of the above. ‖

2. A BRANCHING PROCESS APPLICATION

Consider two Galton Watson branching processes in which individuals at the end of their lifetime give birth to a random number of offspring. Let $X_{jn}^{(i)}$, $j \geq 1$, $n \geq 0$ denote the number of offspring of the jth individual of the nth generation in the ith branching process, $i = 1$, 2 . Suppose that the random variables $X_{jn}^{(i)}$, $j \geq 1$, $n \geq 0$ are independent for $i = 1$, 2 and have a distribution not depending on j . In addition, suppose that

$$X_{jn}^{(1)} \underset{v}{\geq} X_{jn}^{(2)} \quad \text{for all } n , j .$$

Let $Z_n^{(i)}$, $i = 1$, 2 denote the size of the nth generation of the ith process.

Proposition 2:

If $Z_0^{(i)} = 1$, $i = 1$, 2 , then $Z_n^{(1)} \underset{v}{\geq} Z_n^{(2)}$ for all n .

Proof:

The proof is by induction on n . As it is true for $n = 0$, assume it for n . Now,

$$Z_{n+1}^{(1)} = \sum_{j=1}^{Z_n^{(1)}} X_{j,n}^{(1)}$$

$$Z_{n+1}^{(2)} = \sum_{j=1}^{Z_n^{(2)}} X_{j,n}^{(2)}$$

and so the result follows from Theorem 1.||

We now show that if the second (less) variable process has the same mean number of offspring per individual as does the first then it is less likely, at each generation to become extinct.

Corollary 3:

Suppose $E\left[X_{jn}^{(1)}\right] = E\left[X_{jn}^{(2)}\right]$ for all j, n. If $Z_0^{(1)} = 1$, $i = 1, 2$ and $X_{jn}^{(1)} \geq X_{jn}^{(2)}$ for all j, n

$$P\left\{Z_n^{(1)} = 0\right\} \geq P\left\{Z_n^{(2)} = 0\right\} \text{ for all } n.$$

Proof:

From Proposition 2 we have that $Z_n^{(1)} \geq Z_n^{(2)}$ and thus

$$\sum_{i=1}^{\infty} P\left\{Z_n^{(1)} \geq i\right\} \geq \sum_{i=1}^{\infty} P\left\{Z_n^{(2)} \geq i\right\}$$

or, equivalently, since $E\left[Z_n^{(i)}\right] = \sum_{i=0}^{\infty} E\left[X_{ji}^{(i)}\right] = E\left[Z_n^{(i)}\right] = \mu$

$$\mu - P\left\{Z_n^{(1)} \geq 1\right\} \geq \mu - P\left\{Z_n^{(2)} \geq 1\right\}$$

which proves the result. ||

Remark:

(i) In [2], Freedman and Purves showed that among all branching processes for which $P\{X_{jn} = 1\} = 0$ and $E\{X_{jn}\} = M < 2$, all j, n, the one having the least chance of going extinct is the one with $P\{X_{jn} = 0\} = 1 - M/2 = 1 - P\{X_{jn} = 2\}$. This also follows from Corollary 3 upon application of the ... X_{jn} (with $c = 0$).

<u>Lemma 4</u>:

Let $P\{X = 1\} = \alpha$, $P\{X = 0\} = (1 - \alpha) - \dfrac{(M - \alpha)}{2}$, $P\{X = 2\} = \dfrac{M - \alpha}{2}$

and let Y be a nonnegative, integer valued, and such that $P\{Y = 1\} \leq \alpha$

and $E[Y] = M$. If $\alpha < M < 2 - \alpha$, then $X \underset{v}{\leq} Y$.

<u>Proof</u>:

We must show that

$$\sum_{i=n+1}^{\infty} P\{X \geq i\} \leq \sum_{i=n+1}^{\infty} P\{Y \geq i\} \ , \ n = 1,2, \ldots \ .$$

As $E[X] = E[Y] = M$. this is equivalent to

$$\sum_{i=1}^{n} P\{X \geq i\} \geq \sum_{i=1}^{n} P\{Y \geq i\} \ , \ n = 1,2, \ldots \ .$$

When $n = 1$, the above reduces to $P\{X = 0\} \leq P\{Y = 0\}$. This follows

since, as $P\{Y = 1\} \leq P\{X = 1\}$, if $P\{Y = 0\} < P\{X = 0\}$ then it would

not be possible for $E[Y]$ to equal $E[X]$. When $n > 1$, the above is

equivalent to

$$M \geq \sum_{i=1}^{n} P\{Y \geq i\}$$

which follows since $E[Y] = M$.||

3. A SHOCK MODEL APPLICATION

Suppose that shocks occur in accordance with a renewal process having interarrival distribution G and mean μ_G . Each shock gives rise to a nonnegative random damage which, independent of all else, has probability distribution F . The damages are assumed to be additive and we let $D(t)$ denote the damage at time t . That is,

$$D(t) = \sum_{i=1}^{M(t)} X_i$$

where X_i is the damage of the ith shock and $M(t)$ is the number of shocks by t . The system is assumed to fail the first time that $D(t)$ exceeds some constant c . That is, the system fails at time $T_{F,G}$ where

$$T_{F,G} = \min \{t : D(t) > c\} .$$

We will obtain a variability result about $T_{F,G}$ when both F and G are NBUE distributions, where a distribution of a nonnegative random variable X is said to be NBUE (new better than used in expectation) if

$$E[X - t \mid X \geq t] \leq E[X] \quad \text{for all} \quad t \geq 0 .$$

Letting

$$N(c) = \max \{n : X_1 + \ldots + X_n \leq c\} .$$

Then the system will fail at the time of the $N(c) + 1$ shock.

Lemma 5:

If F is NBUE, then

$$N(c) + 1 \underset{v}{\leq} N^*(c) + 1$$

where $N^*(c)$ is a Poisson random variable with mean c/μ_F where $\mu_F = E[X]$.

Proof:

As $N(c)$ is just the number of renewals by time c of a renewal process whose interarrival distribution is NBUE with mean μ_F , the result follows from Theorem 3.17 on page 173 of [1]. $\|$

Proposition 6:

If F and G are both NBUE distributions, then

$$T_{F,G} \underset{v}{\leq} T_{E_1,E_2}$$

where E_1 and E_2 are exponential random variables having the same means as F and G respectively.

Proof:

We can express $T_{F,G}$ by

$$T_{F,G} = \sum_{i=1}^{N(c)+1} Y_i$$

where the Y_i , $i \geq 1$, are the interarrival times between successive shocks. They are thus independent and have distribution G . Now, it is well known that an NBUE distribution G is less variable than an exponential distribution with the same mean and so

$$Y_i \underset{v}{\leq} \epsilon_i \quad \text{when} \quad \epsilon_i \text{ is exponential with mean } \mu_G .$$

The result now follows from Lemma 5 and Theorem 1. $\|$

Remark:

As $T_{E_1,E_2} = \sum_{i=1}^{N^*(c)+1} \varepsilon_i$, it follows upon conditioning on $N^*(c)$ that

$$P\left\{T_{E_1,E_2} \leq x\right\} = \sum_{i=0}^{\infty} e^{-c/\mu_F} \frac{(c/\mu_F)^i}{i!} G_{i+1}(x)$$

where $G_n(x)$ is the gamma distribution with parameters n and $1/\mu_G$ (its mean is $n\mu_G$) . Also, if F and G are NBUE, then from Proposition 6 all of the moments of $T_{F,G}$ are no greater than the corresponding moments of T_{E_1,E_2} . For instance,

$$E[T_{F,G}] \leq E\left[T_{E_1,E_2}\right] = E[(N^*(c) + 1)\mu_G] = (c/\mu_F + 1)\mu_G .$$

4. A POPULATION SURVIVAL MODEL

Consider a population of m individuals each of whom is required to spend exactly one time unit out in the field. For each day i, $i = 1,2, \ldots, m$, there is a random variable Y_i which represents the probability that an individual out on the field on day i will survive. That is, given Y_i, each individual sent out on day i will independently survive with probability Y_i. The Y_i, $i = 1, \ldots, m$ are assumed to be independent and identically distributed random variables for which $P\{0 \leq Y_i \leq 1\} = 1$.

A strategy for the population is a positive integer valued vector $\underline{n} = (n_1, \ldots, n_k)$, $k \leq m$, $\sum_1^k n_i = m$, with the interpretation that n_i individuals are sent out on day i, $i = 1, \ldots, k$. Let $N(\underline{n})$ denote the number of individuals that survive under strategy $\underline{n}$.

Proposition 7:

$$N(\underline{n}) \underset{v}{\geq} N(1,1, \ldots, 1) .$$

Proof:

As the variability ordering is closed under convolution, it clearly suffices to prove that if n individuals are sent out on a given day, then the number of survivors is more variable than it would be if the n individuals were sent out on separate days. Hence, we must show that for any convex function f

$$E\left[\sum_{i=0}^{n} f(i)\binom{n}{i} Y^i (1 - Y)^{n-i}\right] \geq \sum_{i=0}^{n} f(i)\binom{n}{i}[E(Y)]^i [1 - E(Y)]^{n-i} .$$

However, it follows from the following lemma that the function

$$g(p) \equiv \sum_{i=0}^{n} f(i)\binom{n}{i}p^{i}(1 - p)^{n-i}$$

is convex in p and so the above follows from Jensen's Inequality.$||$

Lemma 8:

Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with $P\{X_i = 1\} = p$. If f is a convex function, then for any constant c,

$E\left[f\left(\sum_{1}^{n} X_i + c\right)\right]$ is a convex function of p.

Proof:

The proof is by induction. When $n = 1$ we must show that $pf(c + 1) + (1 - p)f(c)$ is convex in p, which is immediate. Assuming the result for $n - 1$ we then have, upon conditioning on X_n,

$$E\left[f\left(\sum_{1}^{n} X_i + c\right)\right] = pE\left[f\left(\sum_{1}^{n-1} X_i + c + 1\right)\right] + (1 - p)E\left[f\left(\sum_{1}^{n-1} X_i + c\right)\right].$$

Hence if we let

$$g_c(p) = E\left[f\left(\sum_{1}^{n-1} X_i + c\right)\right],$$

we must show that

$$h(p) \equiv pg_{c+1}(p) + (1 - p)g_c(p)$$

is convex. Differentiation with respect to p yields

$$h''(p) = 2(g'_{c+1}(p) - g'_c(p)) + pg''_{c+1}(p) + (1 - p)g''_c(p) .$$

Now, $g_c''(p)$ and $g_{c+1}''(p)$ are nonnegative by the induction hypothesis. Also, as f' is increasing in x , we see that

$$g_a'(p) = E\left[f'\left(\sum_1^{n-1} X_i + a\right)\right]$$

is increasing in a and thus $g_{c+1}'(p) - g_c'(p) \geq 0$. Hence,

$$h''(p) \geq 0$$

which proves the result. ||

<u>Remark</u>:

Since $E[N(\underline{n})] = mE[Y]$ for all strategies $\underline{n}$, it follows, as in Corollary 3, that $P\{N(\underline{n}) = 0\}$ is minimized by the strategy $(1,1, \ldots, 1)$.

5. A POISSON SHOCK MODEL

Suppose that unobserved shocks hit a device in accordance with a Poisson process having rate λ. At some time the device is inspected and if n shocks have occurred by that time a cost $f(n)$ is incurred, where f is an increasing, convex function. There are two inspection plans that can be employed—one of which inspects the device after a random time X and the other after a random time Y, and we are interested in determining which plan leads to a smaller expected cost.

Proposition 9 shows that if X is more variable than Y, then it leads to a greater expected cost.

Proposition 9:

Let $\{N(t) , t \geq 0\}$ be a Poisson process with rate λ and let X and Y be nonnegative random variables that are independent of the Poisson process. Then

$$X \underset{v}{\geq} Y \Rightarrow N(X) \underset{v}{\geq} N(Y) .$$

Proof:

Let f be a convex, increasing function and suppose, without loss of generality, that $f(o) = 0$. Let

$$\Delta(n) = f(n + 1) - f(n)$$

$$H(t) = \sum_{n=0}^{\infty} f(n)(\lambda t)^n/n!$$

and note that $\Delta(n) \geq 0$ and increasing in n by assumption. Now,

$$E[f(N(X))] = \int_0^\infty e^{-\lambda t} H(t) dF_X(t)$$

where F_X is the distribution function of X. Hence, it suffices to show that $g(t) = e^{-\lambda t} H(t)$ is a convex increasing function of t. Now,

$$g'(t) = -\lambda e^{-\lambda t} H(t) + e^{-\lambda t} H'(t)$$

$$= e^{-\lambda t} \sum_{n=0}^\infty [f(n + 1) - f(n)] \lambda^{n+1} t^n / n!$$

$$= \lambda E[\Delta(N(t))] .$$

Now, $\Delta \geq 0$ and so $g' \geq 0$. Also, as $N(t)$ is stochastically increasing in t and Δ is increasing, it follows that $E[\Delta(N(t))]$ is increasing in t and so g is convex. ||

Remark:

Proposition is not true for a general renewal process. It is clearly not true for a deterministic renewal process. (If all interarrivals equal 1 and X uniform $(1.9, 2.1)$, $Y \equiv 2$, then $X \underset{v}{\geq} Y$ but $N(X) \underset{st}{\leq} N(Y)$). Even if the interarrival distribution has a decreasing failure rate, Proposition 9 need not be true. For a counterexample, suppose the interarrival distribution is

$$F(x) = p(1 - e^{-\lambda_1 x}) + (1 - p)(1 - e^{-\lambda_2 x}) , \quad 0 < p < 1 , \quad \lambda_1 \neq \lambda_2 .$$

To compute $E[N(t)]$, imagine that at each renewal a coin, having probability p of landing heads, is flipped. If head appears, the next interarrival is exponential with rate λ_1 and if tails, it is exponential with rate λ_2.

If we let $\Lambda(t) = i$, if the rate at t is λ_i, then $\{\Lambda(t), t \geq 0\}$ is a 2-state Markov chain which leaves state 1 (2) to go to state 2 (1) at an exponential rate $\lambda_1 q$ $(\lambda_2 p)$ where $q = 1 - p$. Also, $P\{\Lambda(0) = 1\} = p$. Hence, it follows (see [4], page 221) that

$$P\{\Lambda(t) = 1\} = pe^{-\bar{\lambda}t} + (1 - e^{-\bar{\lambda}t}) \frac{\lambda_2 p}{\bar{\lambda}}$$

where

$$\bar{\lambda} = \lambda_1 q + \lambda_2 p .$$

Hence,

$$\int_0^t P\{\Lambda(s) = 1\}ds = \frac{pq(\lambda_1 - \lambda_2)}{\bar{\lambda}^2} (1 - e^{-\bar{\lambda}t}) + \frac{\lambda_2 p}{\bar{\lambda}} t .$$

We can now compute $E[N(t)]$ as follows:

$$E[N(t)] = \lambda_1 \int_0^t P\{\Lambda(s) = 1\}ds + \lambda_2 \int_0^t P\{\Lambda(s) = 2\}ds$$

$$= \frac{pq(\lambda_1 - \lambda_2)^2}{\bar{\lambda}^2} (1 - e^{-\bar{\lambda}t}) + \frac{\lambda_1 \lambda_2}{\bar{\lambda}} t .$$

Therefore, $E[N(t)]$ is of the form

$$E[N(t)] = A(1 - e^{-ct}) + Bt , \quad A > 0 , \quad B > 0 , \quad c > 0 .$$

Hence if we let X_ε be uniform on $(1 - \varepsilon , 1 + \varepsilon)$, $0 < \varepsilon < 1$, then

$$E[N(X_\varepsilon)] = A - A \int_{1-\varepsilon}^{1+\varepsilon} \frac{e^{-cx}}{2\varepsilon} \, dx + B$$

$$= A \left[1 + \frac{e^{-c}}{2c\varepsilon} (e^{-c\varepsilon} - e^{c\varepsilon}) \right] + B .$$

Hence, $E[N(X_\varepsilon)]$ decreases in ε when ε is near 1. However, $X_{\varepsilon_1} \underset{v}{\geq} X_{\varepsilon_2}$ whenever $0 < \varepsilon_2 < \varepsilon_1 < 1$.

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